Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/4A**

Further Mathematics

Advanced

PAPER 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over ▶

1·4 : Divisibility Tests

1. **In this question you must show detailed reasoning.**

Without performing any division, explain why $n = 20\,210\,520$ is divisible by 66

(4)

66
6 11
2 3

so must check if 20120520 is divisible by 2, 3 and 11.

if divisible by 2 - last digit is even
if divisible by 3 - sum of digits is divisible by 3
if divisible by 11 - alternating sum of digits is divisible by 11

20210520
ends in 0 so is even $\therefore$ $2 \mid 20210520$ ✓
$2+0+2+1+0+5+2+0 = 12$ $3 \mid 12$ $\therefore$ $3 \mid 20210520$ ✓
$2-0+2-1+0-5+2-0 = 0$ $0 \mid 11$ $\therefore$ $11 \mid 20210520$ ✓

Hence, 20210520 is divisible by 2, 3, 11
$\therefore$ 20210520 is divisible by 66 //

2.1: The Axioms For A Group

2. A binary operation ★ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n| \qquad m, n \in \mathbb{Z}_0^+$$

(a) Explain why $\mathbb{Z}_0^+$ is closed under the operation ★ **(1)**

(b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$ **(2)**

(c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under ★ **(2)**

(d) Determine if $\mathbb{Z}_0^+$ forms a group under ★, giving clear justification for your answer. **(3)**

a. if $m, n \in \mathbb{Z}_0^+$, then $m - n \in \mathbb{Z}$ (note here it is $\mathbb{Z}$ as $(m-n)$ could be -ve. if $m < n$) e.g. $m=1, n=2$ $m-n=-1$ $-1 \notin \mathbb{Z}_0^+$

However $|m-n|$ guarantees no -ve. integer values.

$\therefore \mathbb{Z}_0^+$ closed under ★ //

b. $m \star 0 = |m - 0| = |m| = m$

$0 \star m = |0 - m| = |-m| = m$

Must check both cases to guarantee commutativity

$\therefore$ 0 is the identity. //

c. do $m \star m$

$m \star m = |m - m| = |0| = 0$ identity element

$\therefore$ m is its own self inverse

Question 2 continued

a. Check associativity:

$m \star (n \star p) = (m \star n) \star p$

let: $m = 1, n = 2, p = 3$

$1 \star (2 \star 3)$
$= 1 \star |2-3|$
$= 1 \star |-1|$
$= 1 \star 1$
$= |1-1|$
$= 0$

$(1 \star 2) \star 3$
$= |1-2| \star 3$
$= |-1| \star 3$
$= 1 \star 3$
$= |1-3|$
$= |-2|$
$= 2$

$0 \neq 2$
$1 \star (2 \star 3) \neq (1 \star 2) \star 3$

$\therefore$ Not associative so not a group //

(Total for Question 2 is 8 marks)

1.2 : The Euclidean Algorithm | 1.5 : Solving Congruence Equations

3. (a) Use the Euclidean Algorithm to find integers a and b such that

$$125a + 87b = 1$$

(5)

(b) Hence write down a multiplicative inverse of 87 modulo 125

(1)

(c) Solve the linear congruence

$$87x \equiv 16 \pmod{125}$$

(2)

a. gcd (125, 87)

$125 = 1(87) + 38$
$87 = 2(38) + 11$
$38 = 3(11) + 5$
$11 = 2(5) + 1$
$5 = 5(1)$

gcd (125, 87) = 1

$125 = 1(87) + 38 \rightarrow 38 = 125 - 1(87)$
$87 = 2(38) + 11 \rightarrow 11 = 87 - 2(38)$
$38 = 3(11) + 5 \rightarrow 5 = 38 - 3(11)$
$11 = 2(5) + 1 \rightarrow 1 = 11 - 2(5)$
$5 = 5(1)$

rewrite part (a) algorithm but now make the remainder the subject

$1 = 11 - 2[38 - 3(11)]$ — sub in 5 as {38 − 3(11)} from line ③
$1 = 11 - 2(38) + 6(11)$
$1 = 7(11) - 2(38)$
$1 = 7[87 - 2(38)] - 2(38)$ — sub in 11 as {87 − 2(38)} from line ②
$1 = 7(87) - 14(38) - 2(38)$
$1 = 7(87) - 16(38)$
$1 = 7(87) - 16[125 - 1(87)]$ — sub in 38 as {125 − 1(87)} from line ①
$1 = 7(87) - 16(125) + 16(87)$
$1 = 23(87) - 16(125)$

$125(-16) + 87(23) = 1$ // $a = -16, b = 23$

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Question 3 continued

b. $125a + 87b = 1$

$87b \equiv 1 \text{ mod } 125$

$87(23) \equiv 1 \text{ mod } 125$

$\therefore$ multiplicative inverse : 23

c. $87x \equiv 16 \pmod{125}$ $\times 23$ (multiplicative inverse)

$23 \times 87x \equiv 23 \times 16 \pmod{125}$

$x \equiv 368 \pmod{125}$ simplify

$x \equiv 118 \pmod{125}$ but can also have 368 (mod 125) as final answer

$x \equiv 118 \pmod{125}$

4. Let G be a group of order $46^{46} + 47^{47}$

Using Fermat's Little Theorem and explaining your reasoning, determine which of the following are possible orders for a subgroup of G

(i) 11

(ii) 21

(7)

Order of a subgroup must divide the order of a group - Lagrange's theorem

$\therefore$ must check if 11 and 21 divides $46^{46} + 47^{47}$ by Fermat's Little Theorem

Fermat's little theorem:
- if p is prime and a is any integer then $a^p \equiv 1 \pmod p$
- in the case a is not divisible by p, you can write this result as $a^{p-1} \equiv 1 \pmod p$

e.g. $a^{11-1} = a^{10} \equiv 1 \pmod{11}$

i. $46^{46} + 47^{47} \pmod{11}$

$= (46^{10})^4 (46)^6 + (47^{10})^4 (47)^7 \pmod{11}$

$= (46)^6 + (47)^7 \pmod{11}$ $2 \equiv 46 \pmod{11}$ $3 = 47 \pmod{11}$

$= 2^6 + 3^7 \pmod{11}$ $2^6 = 46^6 \pmod{11}$ $3^7 = 47^7 \pmod{11}$

$= 64 + (3)^4(3)^3 \pmod{11}$ $3^4 \pmod{11} = 81 \pmod{11} = 4 \pmod{11}$

$= 64 + 4(3)^3 \pmod{11}$ $3^3 \pmod{11} = 27 \pmod{11} = 5 \pmod{11}$

$= 64 + (4)(5) \pmod{11}$

$= 64 + 20 \pmod{11}$

$= 84 \pmod{11}$

$= 7 \pmod{11}$

$\therefore 11 \nmid 46^{46} + 47^{47}$ so not possible order for a subgroup //

ii. $21 = 7 \times 3$ so need to check for factors of 7 and 3 using:
$a^6 \equiv 1 \pmod 7$ $a^2 \equiv 1 \pmod 3$

CHECK IF DIVISIBLE BY 3:

$46 \equiv 1 \pmod 3$ $a^n = b^n \pmod m$ where $a,b,m,n \in \mathbb{Z}$ and $m,n > 0$

$46^{46} = 1^{46} \pmod 3 = 1 \pmod 3$

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Question 4 continued

$47 \equiv 2 \pmod 3$

$47^{47} \equiv 2^{47} \pmod 3$

$a^n \equiv b^n \pmod m$ where $a,b,m,n \in \mathbb{Z}$ and $m,n > 0$

since $-1 \equiv 2 \pmod 3$

$-1^{47} = -1 \equiv 2^{47} \pmod 3$

$\therefore -1 \equiv 2 \pmod 3 \equiv 2^{47} \pmod 3$

$46^{46} + 47^{47}$

$1 \pmod 3 + 2 \pmod 3 \equiv 3 \pmod 3 \equiv 0 \pmod 3$

$\therefore 3 \mid 46^{46} + 47^{47}$

CHECK IF DIVISIBLE BY 7:

$46 \equiv 4 \pmod 7$

$46^{46} \equiv 4^{46} \pmod 7$

$a^{7-1} = a^6 \equiv 1 \pmod 7$

$\equiv 4^{6\times 7 + 4} \pmod 7$

$\equiv (4^6)^7 \times 4^4 \pmod 7$

$\equiv 1^7 \times 4^4 \pmod 7$

$\equiv 256 \pmod 7$

$256 \equiv 4 \pmod 7$

$\therefore 46^{46} \equiv 4 \pmod 7$

$47 \equiv 5 \pmod 7$

$47^{47} \equiv 5^{47} \pmod 7$

$\equiv (5^6)^7 \times 5^5 \pmod 7$

$\equiv 1^7 \times 5^5 \pmod 7$

$3 \equiv 3125 \pmod 7$

$3125 \equiv 3 \pmod 7$

$47^{47} \equiv 3 \pmod 7$

$46^{46} + 47^{47}$

$4 \pmod 7 + 3 \pmod 7 \equiv 7 \pmod 7 \equiv 0 \pmod 7$

$\therefore 7 \mid 46^{46} + 47^{47}$

As $46^{46} + 47^{47}$ is divisible by 3 and 7,

$21 \mid 46^{46} + 47^{47}$

$\therefore$ is a possible order for a subgroup //

(Total for Question 4 is 7 marks)

3.1: Loci on an Argand Diagram 3.2: Regions in an Argand Diagram

5. The point P in the complex plane represents a complex number z such that

$$|z+9| = 4|z-12\mathrm{i}|$$

Given that, as z varies, the locus of P is a circle,

(a) determine the centre and radius of this circle.

(6)

(b) Shade on an Argand diagram the region defined by the set

$$\{z \in \mathbb{C} : |z+9| < 4|z-12\mathrm{i}|\} \cap \left\{z \in \mathbb{C} : -\frac{\pi}{4} < \arg\left(z - \frac{3+44\mathrm{i}}{5}\right) < \frac{\pi}{4}\right\}$$

(4)

a. USE ALGEBRAIC APPROACH OF EVALUATING LOCI:

$|z+9| = 4|z-12i|$

let $z = x+iy$

$|x+iy+9| = 4|x+iy-12|$

seperate into real + imaginary parts

$|(x+9)+(y)i| = 4|(x-12)+(y)i|$

$\sqrt{(x+9)^2+(y)^2} = 4\sqrt{(x-12)^2+(y)^2}$

$\left(\sqrt{(x+9)^2+(y)^2}\right)^2 = (4)^2\left(\sqrt{(x-12)^2+(y)^2}\right)^2$ square both sides to get rid of sqrt $\sqrt{\ }$

$(x+9)^2+(y)^2 = 16[(x-12)^2+(y)^2]$

$x^2+18x+81+y^2 = 16[x^2-24x+144+y^2]$

$x^2+18x+81+y^2 = 16x^2-384x+2304+16y^2$

$15x^2-18x+15y^2-384y+2223 = 0$ $\div 15$

$x^2-\frac{6}{5}x+y^2-\frac{128}{5}y+\frac{741}{5} = 0$

$(x-0.6)^2-0.36+(y-12.8)^2-163.84+\frac{741}{5} = 0$ complete the square for both x & y

$(x-0.6)^2+(y-12.8)^2 = 16$ rearrange to get circle eq^n

Centre: $0.6+12.8i$

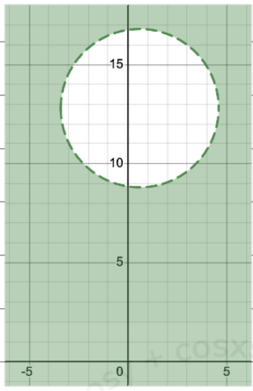
radius: $\sqrt{16} = 4$

Question 5 continued

b. N.B: inequalities are strict (<) so must use dotted lines.

$|z+9| < 4|z-12i|$

↳ shade region containing $(-9,0)$ since it is on ... < side



$-\frac{\pi}{4} < \arg\left(z - \frac{3+44i}{5}\right) < \frac{\pi}{4}$ ⟶ $\left\{-\frac{\pi}{4} < \arg\left(z - \frac{3+44i}{5}\right)\right\} \cap \left\{\arg\left(z - \frac{3+44i}{5}\right) < \frac{\pi}{4}\right\}$

↳ represents a region enclosed by 2 half-lines $\arg\left(z - \frac{3+44i}{5}\right) = -\frac{\pi}{4}$ and $\arg\left(z - \frac{3+44i}{5}\right) = \frac{\pi}{4}$.

↳ plot the point $\left(\frac{3}{5}, \frac{44}{5}\right)$

↳ draw horizontal dotted line from $\left(\frac{3}{5}, \frac{44}{5}\right)$

↳ draw a line @ $\theta = -\frac{\pi}{4}$

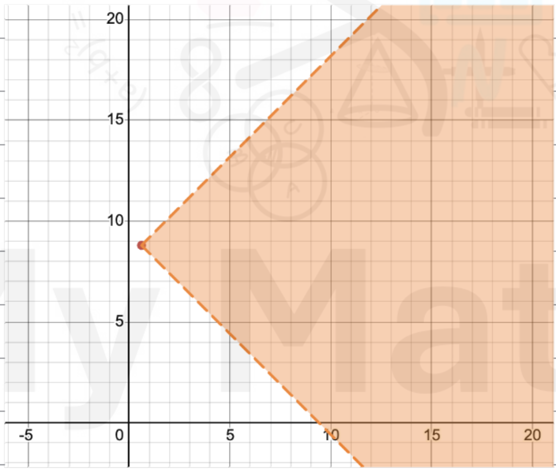
↳ shade region from $\theta = -\frac{\pi}{4}$ to dotted line clockwise

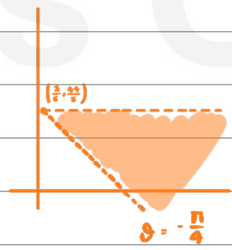
↳ plot the point $\left(\frac{3}{5}, \frac{44}{5}\right)$

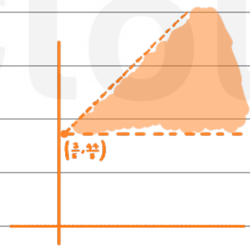
↳ draw horizontal dotted line from $\left(\frac{3}{5}, \frac{44}{5}\right)$

↳ draw a line @ $\theta = \frac{\pi}{4}$

↳ shade region from dotted line to $\theta = \frac{\pi}{4}$ clockwise.

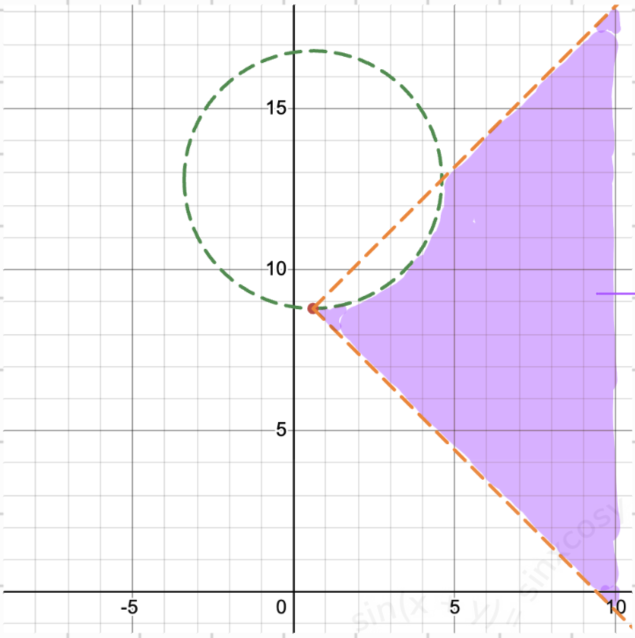






combine

Draw both diagrams on 1 diagram and shade overlapping region:



shaded region which satisfies:

$\{z \in \mathbb{C} : |z+9| < 4|z-12i|\} \cap \{z \in \mathbb{C} : -\frac{\pi}{4} < \arg(z - \frac{3+44i}{5}) < \frac{\pi}{4}\}$

4.4: Proving Closed Forms

6. A recurrence system is defined by

$$u_{n+2} = 9(n+1)^2 u_n - 3u_{n+1} \qquad n \geqslant 1$$

$$u_1 = -3,\ u_2 = 18$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n n!$$

(6)

Basis step:

let $n=1$, $u_1 = (-3)^1 1!$

$u_1 = -3$

let $n=2$, $u_2 = (-3)^2 2!$

$u_2 = (9)(2\times1)$

$u_2 = 18$

$\therefore$ true for $n=1$ and $n=2$

Assumption step:

assume $n=k$ $\quad u_k = (-3)^k k!$

assume $n=k+1$ $\quad u_{k+1} = (-3)^{k+1}(k+1)!$

Inductive step:

$u_{k+2} = 9(k+1)^2\{(-3)^k k!\} - 3\{(-3)^{k+1}(k+1)!\}$

$u_{k+2} = 9(k+1)^2\{(-3)^k k!\} - 3\{(-3)(-3)^k(k+1)(k)!\}$ factor out $(-3)(-3)^k k!$

$u_{k+2} = (-3)(-3)^k k!\,[-3(k+1)^2 - 3(k+1)]$ factor out $(k+1)$

$u_{k+2} = (-3)^{k+1} k!\,[(k+1)\{-3(k+1) - 3\}]$

$u_{k+2} = (-3)^{k+1} k!\,[(k+1)\{-3k-3-3\}]$

$u_{k+2} = (-3)^{k+1} k!\,[(k+1)(-3k-6)]$ factor out 3

$u_{k+2} = (-3)^{k+1} k!\,[(k+1)(-3)(k+2)]$ $\quad k! \times (k+1) \times (k+2) = (k+2)!$ $\quad (-3)^{k+1} \times (-3)^1 = (-3)^{k+2}$

$u_{k+2} = (-3)^{k+2}(k+2)!$ // Result holds for $k+2$

Conclusion:

Hence if true for $n=k$ and $n=k+1$ then true for $n=k+2$. As also true for $n=1$ and $n=2$ then true for all $n \in \mathbb{N}$ by mathematical induction.

6.1: Reduction Formulae | 6.3: Area of a Surface of Revolution

7. **In this question you must show all stages of your working.**

You must not use the integration facility on your calculator.

$$I_n = \int t^n \sqrt{4+5t^2}\,dt \qquad n \geqslant 0$$

(a) Show that, for $n > 1$

$$I_n = \frac{t^{n-1}}{5(n+2)}(4+5t^2)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)} I_{n-2}$$

(5)

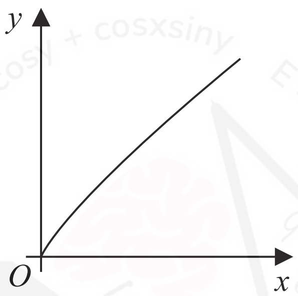


Figure 1

The curve shown in Figure 1 is defined by the parametric equations

$$x = \frac{1}{\sqrt{5}}t^5 \qquad y = \frac{1}{2}t^4 \qquad 0 \leqslant t \leqslant 1$$

use these limits for integral part (b)

This curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

(5)

$$\pi \int_0^1 t^7 \sqrt{4+5t^2}\,dt$$

Using the results in parts (a) and (b) and making each step of your working clear,

(c) determine the value of the external surface area of the shell, giving your answer to 3 significant figures.

(5)

a. $I_n = \int t^n \sqrt{4+5t^2}\,dt$

$I_n = \int t^{n-1}\,t\sqrt{4+5t^2}\,dt$

split $t^n = t^{n-1}\,t$

this is so when applying integration by parts, you can pick $t\sqrt{4+5t^2}$ as v' as this is reverse chain rule

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Question 7 continued

$$I_n = \int \underbrace{t^{n-1}}_{u}\ \underbrace{t\sqrt{4+5t^2}}_{v'}\ dt$$

$u = t^{n-1}$ $\qquad v' = t\sqrt{4+5t^2} = t(4+5t^2)^{1/2}$

$u' = (n-1)t^{n-2}$ $\qquad v = \frac{1}{15}(4+5t^2)^{3/2}$

$\boxed{\text{Integration by parts: } \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \text{ Given in formulae booklet}}$

$$I_n = [t^{n-1}][\tfrac{1}{15}(4+5t^2)^{3/2}] - \int[(n-1)t^{n-2}][\tfrac{1}{15}(4+5t^2)^{3/2}]\,dt$$

they are constants so take them out of the integral

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{1}{15}(n-1)\int t^{n-2}(4+5t^2)^{3/2}\,dt$$

split

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{1}{15}(n-1)\int t^{n-2}(4+5t^2)(4+5t^2)^{1/2}\,dt$$

expand out

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{1}{15}(n-1)\int 4t^{n-2}(4+5t^2)^{1/2} + 5t^2t^{n-2}(4+5t^2)^{1/2}\,dt$$

combine

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{1}{15}(n-1)\int 4t^{n-2}(4+5t^2)^{1/2} + 5t^n(4+5t^2)^{1/2}\,dt$$

split into 2 integrals

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{1}{15}(n-1)\int 4t^{n-2}(4+5t^2)^{1/2}\,dt - \tfrac{1}{15}(n-1)\int 5t^n(4+5t^2)^{1/2}\,dt$$

take 4 out of integral $\qquad$ take 5 out integral

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{4}{15}(n-1)\int t^{n-2}(4+5t^2)^{1/2}\,dt - \tfrac{5}{15}(n-1)\int t^n(4+5t^2)^{1/2}\,dt$$

$$I_n = \tfrac{1}{15}t^{n-1}(4+5t^2)^{3/2} - \tfrac{4}{15}(n-1)I_{n-2} - \tfrac{5}{15}(n-1)I_n$$

rearrange for I_n

$$I_n\{1 + \tfrac{5}{15}(n-1)\} = \frac{t^{n-1}}{15}(4+5t^2)^{3/2} - \frac{4(n-1)}{15}I_{n-2}$$

$$I_n\{\tfrac{2+n}{3}\} = \frac{t^{n-1}}{15}(4+5t^2)^{3/2} - \frac{4(n-1)}{15}I_{n-2}$$

$\times \frac{3}{2+n}$

$$I_n = \frac{3t^{n-1}}{\cancel{15}_5(2+n)}(4+5t^2)^{3/2} - \frac{\cancel{3}\times4(n-1)}{\cancel{15}_5(2+n)}I_{n-2}$$

$$I_n = \frac{t^{n-1}}{5(n+2)}(4+5t^2)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)}I_{n-2} \quad //$$

Question 7 continued

b. surface area of revolution: $S_x = 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ FORMULAE BOOKLET

$x = \frac{1}{\sqrt{5}} t^5$

$\frac{dx}{dt} = \frac{5}{\sqrt{5}} t^4 = \sqrt{5}\, t^4$

$\left(\frac{dx}{dt}\right)^2 = \left(\sqrt{5}\, t^4\right)^2 = 5t^8$

$y = \frac{1}{2} t^4$

$\frac{dy}{dt} = \frac{4}{2} t^3 = 2t^3$

$\left(\frac{dy}{dt}\right)^2 = (2t^3)^2 = 4t^6$

$S_x = 2\pi \int_0^1 \left(\frac{1}{2} t^4\right) \sqrt{(5t^8) + (4t^6)}\, dt$

factor out $\frac{1}{2}$; $2 \times \frac{1}{2} = 1$ outside integral

$S_x = \pi \int_0^1 t^4 \sqrt{(5t^8) + (4t^6)}\, dt$

factor out t^6 in sqrt $\sqrt{\ }$

$S_x = \pi \int_0^1 t^4 \sqrt{t^6(5t^2 + 4)}\, dt$

$\sqrt{t^6(5t^2+4)} = \left[t^6(5t^2+4)\right]^{1/2} = (t^6)^{1/2} (5t^2+4)^{1/2}$

$= t^3 (5t^2+4)^{1/2}$

$= t^3 \sqrt{5t^2+4}$

simplify $\sqrt{t^6(5t^2+4)}$

$S_x = \pi \int_0^1 t^4 \times t^3 \sqrt{5t^2+4}\, dt$

$S_x = \pi \int_0^1 t^7 \sqrt{5t^2+4}\, dt$

$S_x = \pi \int_0^1 t^7 \sqrt{4+5t^2}\, dt$ //

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Question 7 continued

c. must calculate I_7 when $n=7$

$$I_7 = \left[\frac{t^6}{5\times 9}(4+5t^2)^{3/2}\right]_0^1 - \frac{4\times 6}{5\times 9} I_5$$

$$I_5 = \left[\frac{t^4}{5\times 7}(4+5t^2)^{3/2}\right]_0^1 - \frac{4\times 4}{5\times 7} I_3$$

$$I_3 = \left[\frac{t^2}{5\times 5}(4+5t^2)^{3/2}\right]_0^1 - \frac{4\times 2}{5\times 5} I_1$$

work out I_1 directly:

$$I_1 = \int_0^1 t\sqrt{4+5t^2}\,dt = \left[\frac{1}{15}(4+5t^2)^{3/2}\right]_0^1$$

$$= \left[\left\{\frac{1}{15}(4+5(1)^2)^{3/2}\right\} - \left\{\frac{1}{15}(4+5(0)^2)^{3/2}\right\}\right]$$

$$= \frac{19}{15}$$

$$I_1 = \frac{19}{15}$$

$$I_3 = \left[\left\{\frac{(1)^2}{5\times 5}(4+5(1))^{3/2}\right\} - \left\{\frac{(0)^2}{5\times 5}(4+5(0))^{3/2}\right\}\right] - \frac{4\times 2}{5\times 5}\left(\frac{19}{15}\right)$$

$$= \frac{27}{25} - \frac{152}{375}$$

$$= \frac{253}{375}$$

$$I_5 = \left[\left\{\frac{(1)^4}{5\times 7}(4+5(1))^{3/2}\right\} - \left\{\frac{(0)^4}{5\times 7}(4+5(0))^{3/2}\right\}\right] - \frac{4\times 4}{5\times 7}\left(\frac{253}{375}\right)$$

$$= \frac{27}{35} - \frac{4048}{13125}$$

$$= \frac{6077}{13125}$$

$$I_7 = 1\int_0^1 t^7\sqrt{4+5t^2}\,dt$$

however integral we are trying to work out is:

$$\pi\int_0^1 t^7\sqrt{4+5t^2}\,dt$$

$$I_7 = \left[\left\{\frac{(1)^6}{5\times 9}(4+5(1))^{3/2}\right\} - \left\{\frac{(0)^6}{5\times 9}(4+5(0))^{3/2}\right\}\right] - \frac{4\times 6}{5\times 9}\left(\frac{6077}{13125}\right)$$

$$= \frac{27}{45} - \frac{145848}{590625} = 0.3530615873$$

HOWEVER NOTICE INTEGRAL I_n DOESN'T INCLUDE π OUTSIDE INTEGRAL

$\pi \times 0.3530615873 = 1.109175689 \approx 1.11$ units^2 (3s.f.)

(Total for Question 7 is 15 marks)

8.

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector for **A**

(a) (i) determine the eigenvalue corresponding to this eigenvector **(1)**

(ii) hence show that $p = 2$ **(2)**

(iii) determine the remaining eigenvalues and corresponding eigenvectors of **A** **(7)**

(b) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$ **(1)**

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant. **(2)**

With respect to a fixed origin O, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

(ii) determine a general solution for the displacement of the particle. **(4)**

ai. $Ax = \lambda x$

$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

multiply out 1st row and compare coefficients:

$(5)(2) + (-2)(1) + (5)(-2) = 2\lambda$

$-2 = 2\lambda$

$\lambda = -1$

∴ corresponding eigenvalue is -1 //

DO NOT WRITE IN THIS AREA

Question 8 continued

b. multiply out 2^{nd} row to find p:

$(0)(2) + (3)(1) + (p)(-2) = \lambda$ replace λ with -1 as worked in part (ai)

$3 - 2p = -1$

$4 = 2p$

$\boxed{p = 2}$

c. characteristic eq^n: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 2 \\ -6 & 6 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 2 \\ -6 & 6 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & 2 \\ -6 & 6 & -4-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$$\det\left[\begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & 2 \\ -6 & 6 & -4-\lambda \end{pmatrix}\right] = 0$$

$$(5-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 6 & -4-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 0 & 2 \\ -6 & -4-\lambda \end{vmatrix} + (5)\begin{vmatrix} 0 & 3-\lambda \\ -6 & 6 \end{vmatrix} = 0$$

$$(5-\lambda)\left[(3-\lambda)(-4-\lambda) - (6)(2)\right] + (2)\left[\cancel{(0)(-4-\lambda)} - (2)(-6)\right] + (5)\left[\cancel{(0)(6)} + (-6)(3-\lambda)\right] = 0$$

$$(5-\lambda)\left[\lambda^2 + 4\lambda - 3\lambda - 12 - 12\right] + 2\left[12\right] + 5\left[6\lambda - 18\right] = 0$$

$(5-\lambda)(\lambda^2 + \lambda - 24) + 24 + 30\lambda - 90 = 0$

$5\lambda^2 + 5\lambda - 120 - \lambda^3 - \lambda^2 + 24\lambda + 24 + 30\lambda - 90 = 0$

$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

P 6 6 8 0 2 A 0 2 5 2 8

Turn over

Question 8 continued

use factor theorem to find a factor of cubic.

$f(-1): (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$

$\therefore (\lambda+1)$ is a factor.

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \lambda+1 & \lambda^3 - 4\lambda^2 + \lambda + 6 \\ - & \lambda^3 + \lambda^2 \\ & -5\lambda^2 + \lambda \\ - & -5\lambda^2 - 5\lambda \\ & 6\lambda + 6 \end{array}$$

$(\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda+1)(\lambda-3)(\lambda-2) = 0$

$\lambda = -1 \quad \lambda = 3 \quad \lambda = 2$

eigenvalues : $-1, 2, 3$

eigenvector for eigenvalue -1 given in Q.

For eigenvalue 2:

$$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 2 \\ -6 & 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$5x - 2y + 5z = 2x$

$0x + 3y + 2z = 2y$

$-6x + 6y - 4z = 2z$

$3x - 2y + 5z = 0 \quad (1)$

$y + 2z = 0 \quad (2)$

$-6x + 6y - 6z = 0 \quad (3)$

Question 8 continued

rearrange (2)

$y = -2z$

sub into eq^n's (1) and (3)

$3x - 2(-2z) + 5z = 0$

$-6x + 6(-2z) - 6z = 0$

$3x + 4z + 5z = 0$

$-6x - 12z - 6z = 0$

$\left.\begin{array}{l} 3x + 9z = 0 \\ -6x - 18z = 0 \end{array}\right\}$ eq^n's are consistent   (1) ÷ 3   (3) ÷ 6

$x = -3z$

can now form a general eigenvector eq^n in terms of z : $\begin{pmatrix} -3z \\ -2z \\ z \end{pmatrix}$ z can be any no.

e.g. let $z = 1$, corresponding eigenvector : $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ MARK SCHEME USES $z = -1$ $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

For eigenvalue 3:

$$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 2 \\ -6 & 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$5x - 2y + 5z = 3x$

$0x + 3y + 2z = 3y$

$-6x + 6y - 4z = 3z$

Question 8 continued

$2x - 2y + 5z = 0$ ①

$2z = 0$ ②

$-6x + 6y - 7z = 0$ ③

$2z = 0$

$z = 0$

Sub $z = 0$ into ① and ③

$2x - 2y + 5(0) = 0$

$-6x + 6y - 7(0) = 0$

$2x - 2y = 0$ } eqⁿs are $\div 2$

$-6x + 6y = 0$ consistent $\div 6$

$x = y$

Can now form a general eigenvector eqⁿ in terms of x: $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$ x can be any no.

e.g. let $x = 1$, eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

b. $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$ ← uses my eigenvector $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

but mark scheme has: $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$

eigenvalues must correspond to eigenvectors

(Total for Question 8 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS

c. $\dot{u} = ku$

$\frac{du}{dt} = ku$

$\frac{1}{u}\,du = k\,dt$

Seperate variables and integrate both sides

$\int \frac{1}{u}\,du = \int k\,dt$

$\ln|u| = kt + c$

$u = e^{kt+c}$

$u = (e^{kt})(e^{c})$

Split e^{kt+c} to $(e^{kt})(e^{c})$ since e^{c} is a constant can replace as a variable A.

$\boxed{u = Ae^{kt}}$ (mark scheme also accepts $u = e^{kt+c}$)

ii. $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

replace $A = PDP^{-1}$

$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = PDP^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Q states $P^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = PD \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

multiply by P^{-1} on both sides

$P^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \cancel{P^{-1}P}D \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

Q states $P^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}$

$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = D \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

replace D with our value of D worked out in part (b)

$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -u \\ 2v \\ 3w \end{pmatrix}$

equating 1^{st} row:

$\dot{u} = -u$

$\therefore k = -1$ (compare coefficients from part (ci))

use different variables (A, B, C) for (u, v, w)

$\therefore \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix}$

must now convert $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

remember $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\downarrow$ multiply both sides by P

$P\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \cancel{P P^{-1}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$P\begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\downarrow$ replace P with our worked out value of P worked out in part (b)

$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\boxed{\begin{pmatrix} 2Ae^{-t} - 3Be^{2t} + Ce^{3t} \\ Ae^{-t} - 2Be^{2t} + Ce^{3t} \\ -2Ae^{-t} + Be^{2t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

Since mark scheme $P = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ the

gen sol^n is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2Ae^{-t} + 3Be^{2t} + Ce^{3t} \\ Ae^{-t} + 2Be^{2t} + Ce^{3t} \\ -2Ae^{-t} - Be^{2t} \end{pmatrix}$